Adaptive Interest Rate Modelling

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Interest Rate

- Essential for pricing derivatives and hedging corresponding risk.
- A signal of macroeconomic activity.
- Influenced by macroeconomic variables.
- Follow unstable dynamic process.
**Classical One-factor Short Rate Models**

**Vasicek Model**

\[ dr(t) = a\{b - r(t)\} dt + \sigma dW_t \]

**CIR Model**

\[ dr(t) = a\{b - r(t)\} dt + \sigma \sqrt{r(t)} dW_t \]

- \( r_t \geq 0 \)
- Mean reversion
Figure 1: Moving window estimator $\hat{a}$ with window sizes 250, 500 and 750.
Figure 2: Moving window estimator $\hat{b}$ with window sizes 250, 500 and 750
Figure 3: Moving window estimator $\hat{\sigma}$ with window sizes 250, 500 and 750
Extended Short Rate Models

Three strands of extended interest rate models:

- Jump-Diffusion Models
- Regime-Switching Models
- Time-varying Coefficients Models.
Outline

1. Motivation ✓
2. Local Adaptive Approach for CIR model
3. Simulations
4. Empirical Study
5. Conclusion
Model

The time-varying CIR model with $\theta_t = (a_t, b_t, \sigma_t)^\top$:

$$dr(t) = a_t \{b_t - r(t)\} \, dt + \sigma_t \sqrt{r(t)} \, dW_t$$

(1)

Discretization:

$$r_{t_{i+1}} - r_{t_i} = a_t \{b_t - r_{t_i}\} \Delta t + \sigma_t \sqrt{r_{t_i}} Z_i$$

(2)

$$\{Z_i\} \sim N(0, \Delta t)$$
Local Parametric Approach (LPA)

Given time point $t$, go back and split time series into $K$ intervals, $l_0 \subset l_1 \subset \cdots \subset l_K$, with $l_k = [t - m_k + 1, t]$.

- Accept the smallest interval $l_0$ without change point (i.e. homogeneous interval).
- Sequentially check the historical intervals to search a change point.
- Two methods of LPA: Local model selection (LMS) and Local change point (LCP).
Algorithm of LCP

Goal: Find an unknown change point \( \tau \) in the interval \( I_k \).

1. Determine \( \hat{\theta} = \tilde{\theta}_0 \).
2. Increase interval to \( I_k, k \geq 1 \). Get \( \tilde{\theta}_{I_k} \).
3. Compare test statistics with critical value. If test statistic is accepted go to step 4, otherwise go to step 5.
4. Let \( \hat{\theta} = \tilde{\theta}_{I_k} \), and set \( k = k + 1 \), repeat step 2.
5. Detect the change point \( \tau \) in \( I_k, I_{k-1} = I_{k-1} \) without change point.
Why we use LPA?

- Find the longest stable interval for each $t$.
- Allow for structural breaks and jumps in parameter values.
- Distinguish recession and expansion regimes.
Test Statistic

Test statistics $T_{l_{k+1}, \tau}$:

$$T_{l_{k+1}, \tau} = L_J(\tilde{\theta}_J) + L_{J^c}(\tilde{\theta}_{J^c}) - L_{l_{k+1}}(\tilde{\theta}_{l_{k+1}})$$

(3)

where $J = [\tau + 1, t]$, and $J^c = [t - m_{k+1}, \tau]$, and $\tau \in J_k = l_k \setminus l_{k-1}$.

Consider the supremum of the test statistics over interval $J_k$:

$$T_k = \sup_{\tau \in J_k} T_{l_{k+1}, \tau}$$

(4)
Test Algorithm

Figure 4: Construction of the Test Statistics in the Local Change Point Test
The criteria for testing homogeneous intervals:

\[ T_k \leq z_k, \quad \text{for} \quad k \leq \hat{k} \tag{5} \]

and \( T_{k+1} > z_{\hat{k}+1} \).

\( \hat{I}_k \) is the longest time homogeneous interval for time point \( t \),

\( \hat{\theta}_t = \tilde{\theta}_{\hat{I}_k} \).

\( z_k \) is the critical value, obtained by Monte-Carlo simulations.
Risk Bound

Parametric risk bound $R_r(\theta_0)$, given the true value $\theta_0$, for any interval $I_k$,

$$E_{\theta_0} | L_{I_k}(\tilde{\theta}_{I_k}, \theta_0)|^r \leq R_r(\theta_0)$$

(6)

where

$L_{I_k}(\tilde{\theta}_{I_k}, \theta_0) = L_{I_k}(\tilde{\theta}_{I_k}) - L_{I_k}(\theta_0)$ is the likelihood ratio between the two parameters.
Small Modeling Bias Condition

The SMB condition for the interval $l_k$, and given some $\theta \in \Theta$:

$$E \Delta l_k(\theta) \leq \Delta$$

(7)

and

$$\Delta l_k = \sum_{t \in l_k} \mathcal{K}\{r(t), r(t; \theta)\}$$

$\mathcal{K}\{r_t, r_t(\theta)\}$ : Kullback-Leibler divergence between $P_{r(t)}$ and $P_{r(t; \theta)}$.

Oracle Choice $k^*$: the largest $l_k^*$ s.t. (7) holds.
Propagation Condition and Stability

Propagation:
\[ E_{\theta_0} |L_{I_k}(\tilde{\theta}_{I_k}, \hat{\theta}_{I_k})|^r \leq \rho \Re_r(\theta_0) \]  \hspace{1cm} (8)

Stability: \( I_k \) is accepted interval, then \( \hat{\theta}_{I_k} = \tilde{\theta}_{I_k} \)

\[ L(\hat{\theta}_{I_k}, \hat{\theta}_{I_{k+1}}) \leq \delta_k \]  \hspace{1cm} (9)

For fixed \( \Delta \), the loss \( |L_{I_k}(\tilde{\theta}_{I_k}, \theta_0)|^r \) stochastically bounded by a constant proportional to \( e^\Delta \)
Critical Value

- Sequential choice of critical values \( \hat{z}_k \).
- Change point detected at step \( \ell \leq k \).
- \( B_\ell \): rejection at step \( \ell \).

\[
B_\ell = \{ T_1 \leq \hat{z}_1, \ldots, T_{\ell-1} \leq \hat{z}_{\ell-1}, T_\ell > \hat{z}_\ell \}
\]

and \( \hat{\theta}_{l_k} = \tilde{\theta}_{l_{\ell-1}} \) on \( B_\ell \), \( \ell = 1, 2, \ldots, k \).
Critical Value

To determine $z_1$, \[
\max_{k=1,\ldots,K} E_{\theta_0} |L(\tilde{\theta}_{l_k}, \tilde{\theta}_{l_0})| \mathbf{1}(\mathcal{B}_1) \leq \rho \mathcal{R}_r(\theta_0)/K \tag{10}
\]

$\mathcal{B}_\ell$ only depends on $z_1, \ldots, z_\ell$, controlled by $z_\ell$. The minimal value ensures \[
\max_{k \geq l} E_{\theta_0} |L(\tilde{\theta}_{l_k}, \tilde{\theta}_{l_{l-1}})| \mathbf{1}(\mathcal{B}_l) = \rho \mathcal{R}_r(\theta_0)/K \tag{11}
\]

$\mathcal{R}_r(\theta_0)$ is parametric risk bound.
Choice of the Length of Interval

- $l_0$ with length $m_0$.
- Interval $l_k$: $m_k = [m_0 a^k]$ with $a > 1$.
- Results not sensitive to $a$.
- $r=0.5$, power of the loss function.
- $\rho=0.2$, level of the test.
- $m_0 = 40$, $a = 1.25$, and $K=15$, $m_K = 1136$. 
Oracle Property

$l_{k^*}$ is the oracle interval, $E \Delta_{l_{k^*}}(\theta) \leq \Delta$, $k \leq k^*$, and $\hat{\theta}_{l_{k^*}}$ close to the oracle estimate $\tilde{\theta}_{l_{k^*}}$

\[ E \log \left\{ 1 + \frac{|L_{l_{k^*}}(\tilde{\theta}_{l_{k^*}}, \hat{\theta}_{l_{k}})|^r}{R_r(\theta)} \right\} \leq \rho + \Delta \]  \hspace{1cm} (12)

For $k > k^*$, the adaptive estimator $\hat{\theta}_{l_{\hat{k}}}$ satisfies

\[ E \log \left\{ 1 + \frac{|L_{l_{k^*}}(\tilde{\theta}_{l_{k^*}}, \hat{\theta}_{l_{\hat{k}}})|^r}{R_r(\theta)} \right\} \leq \rho + \Delta + \log \left\{ 1 + \frac{\delta_{k^*}^r}{R_r(\theta)} \right\} \] \hspace{1cm} (13)
Simulation Setup

We simulate CIR process with 1500 observations and 100 times.

<table>
<thead>
<tr>
<th>t</th>
<th>a</th>
<th>b</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>t ∈ [1, 500]</td>
<td>0.2</td>
<td>0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>t ∈ [501, 1000]</td>
<td>0.5</td>
<td>0.06</td>
<td>0.1</td>
</tr>
<tr>
<td>t ∈ [1001, 1500]</td>
<td>0.8</td>
<td>0.01</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Table 1: The parameter settings for simulations of the CIR process
Figure 5: LPA estimator $\hat{a}$ and $\hat{b}$ with simulated CIR paths.
Figure 6: LPA estimator $\hat{\sigma}$ and selected time homogeneous intervals.
Data

Yield of 3M US T-Bill from the Federal Reserve Bank of St. Louis from 19980102 to 20090513.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>SD</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_t$</td>
<td>0.0319</td>
<td>0.0176</td>
<td>-0.1159</td>
<td>-1.4104</td>
</tr>
<tr>
<td>$dr_t$</td>
<td>$-1.764 \times 10^{-5}$</td>
<td>0.0006</td>
<td>-0.7467</td>
<td>34.4856</td>
</tr>
</tbody>
</table>

Table 2: Statistical Summary of 3-month T-Bill
Figure 7: 3-month Treasury Bill Rate: 19980102—20090513. Top panel: Daily yields. Bottom panel: Changes of daily yields.
MLE Estimator of CIR model

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>$\hat{a}$</th>
<th>$\hat{b}$</th>
<th>$\hat{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>19980102–20090513</td>
<td>0.2657</td>
<td>0.0153</td>
<td>0.0944</td>
</tr>
<tr>
<td>19980102–20070731</td>
<td>0.1424</td>
<td>0.0252</td>
<td>0.0428</td>
</tr>
<tr>
<td>20070801–20090513</td>
<td>3.6792</td>
<td>0.0081</td>
<td>0.2280</td>
</tr>
</tbody>
</table>

Table 3: Estimated parameters of CIR model using MLE
Critical Value

Figure 8: Critical values with $m_0 = 40, K=15$
Homogeneous Intervals

Figure 9: Selected time homogeneous intervals with $\rho = 0.2$, and $r = 0.5$
Estimator $\hat{a}$

Figure 10: Estimated $\hat{a}$ by LPA
Estimator $\hat{b}$

Figure 11: Estimated $\hat{b}$ by LPA
Estimator $\hat{\sigma}$

Figure 12: Estimated $\hat{\sigma}$ by LPA
In Sample Fitting

Figure 13: Confidence Interval (Red); Real Data (Black); LPA CIR (Blue); CIR (Purple)
Adaptive Interest Rate
Forecasting

$\mathcal{H}$: the prediction period horizon, then the absolute prediction error (APE):

$$APE(t) = \sum_{h \in \mathcal{H}} |r_{t+h} - \hat{r}_{t+h|t}| / |\mathcal{H}|$$

(14)

APE Ratio = $\frac{APE_{LPA}(t)}{APE_{MW}(t)}$

LPA: Local Parametric Approach.
MW: Moving Window Estimation.
Forecasting

Figure 14: The APE ratio between LPA and MW with window size 250. Left: 1-day ahead forecasting; Right: 10-day ahead forecasting.
Performance of Forecasting

<table>
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<tr>
<th>Horizon</th>
<th>MAE</th>
<th>MAE</th>
<th>MAE</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$l = 250$</td>
<td>$l = 500$</td>
<td>$l = 750$</td>
</tr>
<tr>
<td>One Day</td>
<td>LPA $4.74 \times 10^{-4}$</td>
<td>$4.85 \times 10^{-4}$</td>
<td>$4.96 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>MW $4.78 \times 10^{-4}$</td>
<td>$4.41 \times 10^{-4}$</td>
<td>$4.16 \times 10^{-4}$</td>
</tr>
<tr>
<td>Ten Days</td>
<td>LPA 0.0201</td>
<td>0.0215</td>
<td>0.0232</td>
</tr>
<tr>
<td></td>
<td>MW 0.1868</td>
<td>1.0032</td>
<td>1.8054</td>
</tr>
</tbody>
</table>

Table 4: The MAE of 1 day and 10 days ahead forecasting of the short rate based on the LPA and MW.
Conclusion

- Interest rate in recession is more volatile.
- The selected time homogeneous intervals cannot last long due to the complexities of macroeconomy.
- The LPA can detect jumps and structural break points in the interest rate dynamics.
- The LPA outperforms the moving window estimation especially in long horizon forecasting.
References

- P. Čížek, W. Härdle and V. Spokoiny
  Adaptive Pointwise Estimation in Time-inhomogeneous Conditional Heteroscedasticity Models

- C. Cox, E. Ingersoll and A. Ross
  A Theory of the Term Structure of Interest Rates

- J. Franke, W. Härdle and C. Hafner
  Springer Verlag, 2011.
References

J. Hull and A. White
Pricing Interest-Rate-Derivative Securities
Review of Financial Studies Vol.3(1990), No.4, pp. 573-592

V. Spokoiny
Multiscale Local Change Point Detection with Applications to Value-at-Risk

O. Vasicek
An Equilibrium Characterization of the Term Structure
Adaptive Interest Rate Modelling

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